

Energy estimate for initial data on a characteristic cone

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Abstract

The Einstein equations in wave map gauge are a geometric second order system for a Lorentzian metric. To study existence of solutions of this hyperbolic quasi diagonal system with initial data on a characteristic cone which are not zero in a neighbourhood of the vertex one can appeal to theorems due to Cagnac and Dossa, proved for a scalar wave equation, for initial data in functional spaces relevant for their proofs. It is difficult to check that the initial data that we have constructed as solutions of the Einstein wave-map gauge constraints satisfy the more general of the Cagnac-Dossa hypotheses which uses weighted energy estimates. In this paper we start a new study of energy estimates using on the cone coordinates adapted to its null structure which are precisely the coordinates used to solve the constraints, following work of Rendall who considered the Cauchy problem for Einstein equations with data on two intersecting characteristic surfaces.

1 Introduction

In recent work (see summary in [3]) we have considered the Cauchy problem for the Einstein equations with data on a characteristic cone. We have used a wave-map gauge with target a Minkowski metric which admits this cone as a null cone and derived explicit formulae for the constraint equations on initial data, that is the trace on the cone of the looked for Lorentzian metric. These constraints were proved to be necessary and sufficient conditions for a solution of the Einstein equations in wave gauge taking these initial data to be a solution of the original Einstein equations. We have constructed solutions of the constraints which tend to Minkowskian values at the vertex of the cone, but are not necessarily identical to the trace of a Minkowski metric in a neighbourhood of this vertex, as was assumed in a recent book by Christodoulou [4] and a subsequent paper by Klainerman and Rodnianski [8]. The Einstein equations in wave map gauge are a geometric second order system for a Lorentzian metric. To study existence of solutions of this hyperbolic quasi diagonal system with initial data on a characteristic cone which are not zero in a neighbourhood of the vertex we have appealed to a theorem due to Cagnac and Dossa, proved for a scalar wave equation, for initial data in functional spaces relevant for their proofs. However it is difficult to check that the initial data that we have constructed as solutions of the Einstein wave-map gauge constraints satisfy the more general of the Cagnac-Dossa hypotheses which appeals to weighted energy estimates.

In this paper we start a new study of energy estimates using on the cone coordinates adapted to its null structure which are precisely the coordinates we used to solve the constraints, inspired by work of Rendall [10] and Damour and Schmidt [5] who considered the Cauchy problem for Einstein equations with data on two intersecting characteristic surfaces.

2 Definitions and notations

We consider a linear quasidiagonal second order system on a manifold V

$$g^{\alpha\beta}D_{\alpha\beta}^2h = f \tag{2.1}$$

where g is a Lorentzian metric, h and f are sections of a vector bundle \mathcal{V} over V (for example covariant symmetric 2 tensor fields) and D is a covariant derivative in a given metric not necessarily equal to g .

We take for V an open set of R^{n+1} and denote by $y^\alpha, \alpha = 0, i$, with $i = 1, \dots, n$, coordinates admissible for the differential structure of V . We consider a cone C_O of vertex $O \in V$ which has in the coordinates y^α the same equation as the Minkowski cone in standard coordinates

$$y^0 = r, \quad r^2 := \sum_{i=1}^n (y^i)^2; \quad (2.2)$$

we suppose C_O to be a characteristic cone of the Lorentzian metric g : it is well known that the use of normal geodesic coordinates centered at the vertex O shows that the choice (2.2) is no restriction on g and C_O if V is a small enough neighbourhood of O . Cagnac and Dossa use the same representation of a characteristic cone with vertex O . They denote, as we will do, by Y_O^T the future of O limited by $y^0 \leq T$, that is:

$$Y_0^T := \{r \leq y^0 \leq T\} \quad \text{and set} \quad S_t := \{r \leq y^0 = t\}, \quad C_0^T := \{r = y^0 \leq T\}; \quad (2.3)$$

they take as coordinates on C_O the n variables y^i . Of course the cone is not diffeomorphic to R^n , being singular for $\vec{y} := (y^1, \dots, y^n) = 0$.

We define coordinates in V , singular at O , adapted to the null structure of C_O , defined by

$$y^0 = x^1 - x^0, \quad r = x^1, \quad y^i = r\Theta^i(x^A), \quad \text{with} \quad \sum_{i=1}^n (\Theta^i)^2 = 1, \quad (2.4)$$

$x^A, A = 2, \dots, n$ local coordinates on the sphere S^{n-1} . Components of geometric objects in y coordinates are underlined, components are in x coordinates if not underlined.

In the coordinates x^α the equation of C_O is $x^0 = 0$. Traces on the cone are overlined, $\bar{g}^{00} \equiv 0$. The lines $x^A = \text{constant}$ on C_O are geodesic null rays, hence $\bar{g}_{11} \equiv \bar{g}_{1A} \equiv 0$; that is, the trace on C_O of the metric g takes the form

$$\bar{g} = \bar{g}_{00}(dx^0)^2 + 2\nu_0 dx^0 dx^1 + 2\nu_A dx^0 dx^A + \tilde{g}, \quad \tilde{g} := \bar{g}_{AB} dx^A dx^B. \quad (2.5)$$

3 Stress energy tensor

To have norms for tensors on $V \subset R^{n+1}$ we endow it with the euclidean metric

$$\mathbf{e} \equiv (dy^0)^2 + \sum_{i=1, \dots, n} (dy^i)^2, \quad (3.1)$$

which reads in the x^α coordinates

$$\mathbf{e} \equiv (dx^0)^2 - 2dx^1 dx^0 + 2(dx^1)^2 + (x^1)^2 s_{AB} dx^A dx^B.$$

In the x coordinates it holds that

$$\bar{e}^{00} = 2, \quad \bar{e}^{11} = 1, \quad \bar{e}^{01} = 1, \quad \bar{e}^{1A} = \bar{e}^{0A} = 0, \quad \bar{e}^{AB} \equiv (x^1)^{-2} s^{AB}. \quad (3.2)$$

We denote by D the covariant derivative in the metric \mathbf{e} on R^{n+1} , it coincides with the covariant derivative in the Minkowski metric η ,

$$\eta \equiv -(dy^0)^2 + \sum_{i=1, \dots, n} (dy^i)^2 \equiv -(dx^0)^2 + 2dx^1 dx^0 + (x^1)^2 s_{n-1}, \quad (3.3)$$

both these covariant derivatives coinciding with ordinary partial derivatives in the y coordinates.

Indices are raised with the contravariant associate of g . We denote by an underlined dot the pointwise scalar product relative to \mathbf{e} .

Definition 1 *The stress energy tensor of a tensor h is the symmetric 2-tensor:*

$$U^{\alpha\beta} = D^\alpha h_\cdot D^\beta h - \frac{1}{2} g^{\alpha\beta} D_\lambda h_\cdot D^\lambda h. \quad (3.4)$$

We consider a past oriented timelike vector X . The energy momentum vector is

$$\mathcal{P}^\alpha := U^{\alpha\beta} X_\beta. \quad (3.5)$$

The \mathbf{e} -divergence of \mathcal{P} is

$$D_\alpha \mathcal{P}^\alpha \equiv D_\alpha (U^{\alpha\beta} X_\beta) \equiv X_\beta D_\alpha U^{\alpha\beta} + U^{\alpha\beta} D_\alpha X_\beta \quad (3.6)$$

We have

$$D_\alpha U^{\alpha\beta} \equiv g^{\alpha\lambda} D_{\alpha\lambda}^2 h_\cdot D^\beta h + F^\beta. \quad (3.7)$$

with

$$F^\beta \equiv D_\alpha g^{\alpha\lambda} D_\lambda h_\cdot D^\beta h + D^\alpha h_\cdot D_\alpha D^\beta h - \frac{1}{2} D_\alpha (g^{\alpha\beta} g^{\lambda\mu}) D_\lambda h_\cdot D_\mu h - g^{\alpha\beta} D_\alpha D_\lambda h_\cdot D^\lambda h. \quad (3.8)$$

Changing ordering and names of indices we find

$$F^\beta \equiv D^\alpha h_{\underline{z}}(D_\alpha D^\beta h - D^\beta D_\alpha h) + D_\alpha g^{\alpha\lambda} D_\lambda h_{\underline{z}} D^\beta h - \frac{1}{2} D_\alpha (g^{\alpha\beta} g^{\lambda\mu}) D_\lambda h_{\underline{z}} D_\mu h; \quad (3.9)$$

the nullity of the Riemann tensor of the Minkowski metric implies

$$(D_\alpha D^\beta h - D^\beta D_\alpha h \equiv g^{\beta\lambda} (D_\alpha D_\lambda h - D_\lambda D_\alpha h) + D_\alpha g^{\beta\lambda} D_\lambda h \equiv D_\alpha g^{\beta\lambda} D_\lambda h.$$

Finally we see that F^β reduces to the following quadratic form in the derivatives of g

$$F^\beta \equiv \frac{1}{2} D_\alpha (g^{\alpha\mu} g^{\beta\lambda} + g^{\beta\mu} g^{\alpha\lambda} - g^{\alpha\beta} g^{\lambda\mu}) D_\lambda h_{\underline{z}} D_\mu h. \quad (3.10)$$

4 Energy equality

We assume that the contravariant associate of g and X are C^1 in V . Then $\mathcal{P}^\alpha \equiv U^{\alpha\beta} X_\beta \in C^1$ if $h \in C^2$.

When h is solution of the system (2.1) we deduce from (3.6) the equality

$$D_\alpha \mathcal{P}^\alpha = X_\beta (D^\beta h_{\underline{z}} f + F^\beta) + U^{\alpha\beta} D_\alpha X_\beta. \quad (4.1)$$

We denote by $\Omega_{\mathbf{e}}$ the $n+1$ volume form of \mathbf{e} ; it reads in arbitrary coordinates z^α

$$\Omega_{\mathbf{e}} = (\det \mathbf{e}_z)^{\frac{1}{2}} dz^0 \wedge dz^1 \dots \wedge dz^n. \quad (4.2)$$

In the coordinates respectively y^α and x^α it holds that

$$(\det \mathbf{e}_y)^{\frac{1}{2}} \equiv 1, \quad (\det \mathbf{e}_x)^{\frac{1}{2}} \equiv (x^1)^{n-1} |\det s_{n-1}|^{\frac{1}{2}}. \quad (4.3)$$

We recall the identity (d denotes the exterior derivative and a dot the contraction in the metric g)

$$D \cdot \mathcal{P} \Omega_{\mathbf{e}} \equiv d(\mathcal{P} \cdot \omega), \quad (4.4)$$

where ω is the covariant vector valued Leray n form whose components are given in arbitrary coordinates z^α by

$$\omega_\alpha = (-1)^\alpha |\det \mathbf{e}_z|^{\frac{1}{2}} dz^0 \wedge dz^1 \dots \wedge d\hat{z}^\alpha \wedge \dots \wedge dz^n. \quad (4.5)$$

The notation $\hat{\alpha}$ means that the corresponding differential does not appear in the component ω_α .

We choose for X the past oriented vector with components in the y coordinates (recall that we underline such components)

$$\underline{X}_\beta := \delta_\beta^0. \quad (4.6)$$

We integrate with respect to the volume form Ω_e the equality (4.1) on Y_O^T oriented by the natural orientation of R^n and increasing $t := y^0$. The result reads in the y coordinates

$$\int_{Y_O^T} D.\mathcal{P}\Omega_e = \int_0^T \int_{S_t} (\underline{D^0 h}.f + \underline{F^0})\mu_e dt. \quad (4.7)$$

On the other hand, the following identity holds if the integral on its right hand side exists,

$$\int_{Y_O^T} D.\mathcal{P}\Omega_e \equiv \int_{\partial Y_O^T} \mathcal{P}.\omega. \quad (4.8)$$

We have, using the definitions 2.3

$$\partial Y_O^T \equiv S_T \cup C_O^T. \quad (4.9)$$

4.1 Integral on S_T

We have

$$\int_{S_T} \mathcal{P}.\omega = \int_{r \leq T} \underline{\mathcal{P}^0}(T, \vec{y}) dy^1 \dots dy^n, \quad \vec{y} := (y^1, \dots, y^n). \quad (4.10)$$

With the choice we have made of X , $\underline{\mathcal{P}^0}$ reads

$$\underline{\mathcal{P}^0} \equiv \delta_\beta^0 \underline{U^{0\beta}} = \underline{D^0 h}.\underline{D^0 h} - \frac{1}{2} \underline{g^{00}} D_\lambda h.\underline{D^\lambda h}, \quad (4.11)$$

i.e.

$$\underline{\mathcal{P}^0} \equiv \underline{D^0 h}.\underline{D^0 h} - \frac{1}{2} \underline{g^{00}} \left(\underline{g_{00}} \underline{D^0 h}.\underline{D^0 h} + 2 \underline{g_{0i}} \underline{D^0 h}.\underline{D^i h} + \underline{g_{ij}} \underline{D^i h}.\underline{D^j h} \right). \quad (4.12)$$

It is a positive definite quadratic form of \underline{Dh} if g is a Lorentzian metric regularly sliced (see [2, appendix 7]) by S_t .

4.2 Integral on C_O^T

We write the integral on C_O^T in the x^α coordinates. Recalling that $\bar{\mathcal{P}}^0$ denotes the value on C_O of the component with index zero in the x^α coordinates of the vector \mathcal{P} , we find

$$\int_{C_O^T} \mathcal{P} \cdot \omega \equiv \int_0^T \int_{S^{n-1}} \bar{\mathcal{P}}^0(x^1)^{n-1} |\det s_{n-1}|^{\frac{1}{2}} dx^2 \dots dx^n dx^1. \quad (4.13)$$

The components of X in the coordinates x^α are

$$X_\alpha := \underline{X}_\beta \frac{\partial y^\beta}{\partial x^\alpha}, \quad \text{i.e.} \quad X_0 = -1, \quad X_1 = 1, \quad X_A = 0. \quad (4.14)$$

Hence on C_O^T it holds that

$$\bar{\mathcal{P}}^0(x^1, x^A) \equiv -\bar{U}^{00}(x^1, x^A) + \bar{U}^{01}(x^1, x^A), \quad (4.15)$$

where, using previous notations and recalling that

$$\bar{g}^{00} = \bar{g}^{0A} = 0, \quad \nu^0 := \bar{g}^{01} = \frac{1}{\nu_0}, \quad \bar{g}^{A1} \equiv -\nu^0 \nu^A, \quad \bar{g}^{11} \equiv -(\nu^0)^2 \bar{g}_{00} + (\nu^0)^2 \nu^A \nu_A,$$

$$\bar{U}^{00} \equiv (\nu^0)^2 \bar{U}_{11}, \quad \bar{U}^{01} \equiv \nu^0 (\nu^0 \bar{U}_{01} - \nu^0 \nu^A \bar{U}_{1A} + \bar{g}^{11} \bar{U}_{11}), \quad (4.16)$$

hence

$$\bar{\mathcal{P}}^0 \equiv \nu^0 \{(-\nu^0 + \bar{g}^{11}) \bar{U}_{11} + \nu^0 \bar{U}_{01} - \nu^0 \nu^A \bar{U}_{1A}\}, \quad (4.17)$$

with, since $\bar{g}_{11} = \bar{g}_{1A} = 0$,

$$\bar{U}_{1A} \equiv D_A \bar{h}_\perp D_1 \bar{h}, \quad \bar{U}_{11} \equiv D_1 \bar{h}_\perp D_1 \bar{h} \quad (4.18)$$

and

$$\bar{U}_{01} \equiv \overline{D_0 \bar{h}_\perp D_1 \bar{h}} - \frac{1}{2} \nu_0 \overline{D_\lambda \bar{h}_\perp D^\lambda \bar{h}}. \quad (4.19)$$

We have

$$\overline{D_\lambda \bar{h}_\perp D^\lambda \bar{h}} \equiv 2\nu^0 (\overline{D_0 \bar{h}_\perp D_1 \bar{h}} - \nu^A D_A \bar{h}_\perp D_1 \bar{h}) + \bar{g}^{11} D_1 \bar{h}_\perp D_1 \bar{h} + \bar{g}^{AB} D_A \bar{h}_\perp D_B \bar{h},$$

hence the transversal derivative $\overline{D_0 \bar{h}}$ disappears in \bar{U}_{01} which reads

$$\bar{U}_{01} \equiv \nu^A D_A \bar{h}_\perp D_1 \bar{h} - \frac{1}{2} \nu_0 (\bar{g}^{11} D_1 \bar{h}_\perp D_1 \bar{h} + \bar{g}^{AB} D_A \bar{h}_\perp D_B \bar{h}); \quad (4.20)$$

$\bar{\mathcal{P}}^0$ simplifies to the quadratic form

$$\begin{aligned} \bar{\mathcal{P}}^0 \equiv & \nu^0 \{ (-\nu^0 + \bar{g}^{11}) D_1 \bar{h}_z D_1 \bar{h} + \nu^0 \nu^A D_A \bar{h}_z D_1 \bar{h} \\ & - \frac{1}{2} (\bar{g}^{11} D_1 \bar{h}_z D_1 \bar{h} + \bar{g}^{AB} D_A \bar{h}_z D_B \bar{h}) - \nu^0 \nu^A D_A \bar{h}_z D_1 \bar{h} \}, \end{aligned} \quad (4.21)$$

which simplifies to

$$\bar{\mathcal{P}}^0 \equiv -\{ \nu^0 (\nu^0 - \frac{1}{2} \bar{g}^{11}) D_1 \bar{h}_z D_1 \bar{h} + \frac{1}{2} \bar{g}^{AB} D_A \bar{h}_z D_B \bar{h} \}. \quad (4.22)$$

We remark using the values (4.14) of \bar{X}_α that on C_O

$$\bar{g}^{\alpha\beta} \bar{X}_\alpha \bar{X}_\beta \equiv -2\nu^0 + \bar{g}^{11} \quad (4.23)$$

which is negative if \bar{X} is timelike. Hence $\bar{\mathcal{P}}^0 \leq 0$, as foreseen from the general theory since the boundary C_O^T of Y_O^T is null and outgoing. (See [2, appendix 7].)

4.3 Energy equality

We have proved, under the indicated condition, the following theorem.

Theorem 2 *If the metric g is C^1 a C^2 solution of the equation (2.1) satisfies the equality*

$$\begin{aligned} \int_{S_T} \underline{\mathcal{P}}^0(T, \vec{y}) dy^1 \dots dy^n &= - \int_{C_O^T} \bar{\mathcal{P}}^0(x^1, x^A) (x^1)^{n-1} dx^1 \mu_{S^{n-1}} + \\ &\int_0^T \int_{S_t} (\underline{D}^0 h.f + \underline{F}^0)(t, y^1 \dots, y^n) dy^1 \dots dy^n dt. \end{aligned} \quad (4.24)$$

5 Energy inequality

The hypothesis that the Lorentzian metric g is regularly sliced on Y_O^T implies that there exist numbers $C_m > 0$ and $C_M \geq C_m$ such that, with $\vec{y} := (y^i, i = 1, \dots, n)$,

$$C_m \varepsilon(t, \vec{y}) \leq \underline{\mathcal{P}}^0(t, \vec{y}) \leq C_M \varepsilon(t, \vec{y}), \quad (5.1)$$

with

$$\varepsilon(t, \vec{y}) \equiv \left\{ \frac{\partial h}{\partial t} \cdot \frac{\partial h}{\partial t} + \delta^{ij} \frac{\partial h}{\partial y^i} \cdot \frac{\partial h}{\partial y^j} \right\}(t, \vec{y}). \quad (5.2)$$

We set

$$E(t) \equiv \int_{0 \leq r \leq t} \varepsilon_t(t, \vec{y}) dy^1 \dots dy^n, \quad r := \{\Sigma(y^i)^2\}^{\frac{1}{2}}. \quad (5.3)$$

We denote generically by C a number depending only on n and the uniform slicing hypotheses, i.e. C_m and C_M . We have

$$E(t) \leq C \int_{S_t} \mathcal{P}^0 dy^1 \dots dy^n. \quad (5.4)$$

We assume that there exists a continuous function, $C_{Dg}(t)$, of $t \in [0, T]$ such that

$$\sup_{S_t} |Dg| \leq C_{Dg}(t). \quad (5.5)$$

We denote by $C_{|Dg|}$ any number depending only on the uniform slicing bounds of g and the supremum of $C_{Dg}(t)$ for $0 \leq t \leq T$.

Theorem 3 (*energy inequality*) *If g is C^1 and uniformly sliced on Y_O^T any C^2 solution of the equation (2.1) satisfies an inequality*

$$E(T) \leq C e^{C_{|Dg|} T} \int_0^T \left(\|f\|_{L^2(S_t)}^2 + t^{n-1} \int_{S^{n-1}} |\bar{\mathcal{P}}^0(t, x^A)| \mu_{S^{n-1}} \right) dt.$$

Proof. We deduce from (5.1), (5.2) that we have on S_t

$$|D^0 u_{\underline{f}} + \underline{F}^0| \leq C \varepsilon(t)^{\frac{1}{2}} |f| + C_{|Dg|} \varepsilon(t). \quad (5.6)$$

On the other hand we have

$$- \int_{C_O^T} \bar{\mathcal{P}}^0(x^1, x^A) (x^1)^{n-1} dx^1 \mu_{S^{n-1}} \equiv \int_0^T \Phi_t dt, \quad (5.7)$$

with (recall that $x^1 = t$ on C_O)

$$\Phi_t := -t^{n-1} \int_{S^{n-1}} \bar{\mathcal{P}}^0(t, x^A) \mu_{S^{n-1}} \geq 0. \quad (5.8)$$

The equality (5.7) implies the inequality

$$E(T) \leq \int_0^T \{C_{|Dg|} E(t)\} dt + C \int_0^T (\|f\|_{L^2(S_t)}^2 + \Phi_t) dt, \quad (5.9)$$

with

$$\|f\|_{L^2(S_t)}^2 := \int_{0 \leq r \leq t} |f(t, \vec{y})|^2 dy^1 \dots dy^n.$$

By the Gronwall lemma the inequality (5.9) verified by a C^2 solution of (2.1) such that $E(0) = 0$ implies that

$$E(t) \leq Z(t),$$

with $Z(t)$ solution of the differential equation

$$Z'(t) = C_{|Dg|} Z(t) + C(\|f\|_{L^2(S_t)}^2 + \Phi_t) \quad \text{with} \quad Z(0) = 0. \quad (5.10)$$

We look for a solution of (5.10) vanishing for $t = 0$ under the form $Z(t) = k e^{C_{|Dg|} t}$, we find

$$k' e^{C_{|Dg|} t} = C(\|f\|_{L^2(S_t)}^2 + \Phi_t),$$

$$Z(t) \equiv e^{C_{|Dg|} t} \int_0^t e^{-C_{|Dg|} t} C(\|f\|_{L^2(S_t)}^2 + \Phi_t) dt.$$

■

5.1 Uniqueness theorem

A uniqueness theorem for the linear equation (2.1) results immediately from the inequality (5.9) which implies $E(t) \equiv 0$ if $f \equiv \Phi_t = 0$. We state:

Theorem 4 *Two C^2 solutions of the equation (2.1) in Y_O^T with g a C^1 Lorentzian metric uniformly sliced coincide in Y_O^T if they have the same trace on C_O^T .*

6 Open problems

The energy inequality can very likely be extended to tensors which are in spaces obtained by completion of C^2 using norms which appear in this inequality.

One could perhaps, using the energy inequality and some functional analysis, prove an existence theorem for a generalized solution of the linear system, as one does for a linear system with spacelike Cauchy data, though one should probably for such a proof use a double null foliation, like in Klainerman and Nicolò [7].

Anyway one needs higher order estimates to have results in the case of quasilinear equations. In the case of a cone as support of the initial data a problem for the use of standard embedding and multiplication properties of Sobolev spaces is that the sections S_t cannot be considered as Riemannian manifolds with equivalent Sobolev constants when t tends to zero, the vertex of the cone. A remedy proposed by Dossa in the case of a scalar equation is to use the y^i as coordinates on the cone and to scale \overrightarrow{y} by powers of t^{-1} in order to work in a fixed sphere of R^n . We postpone the application of this idea to a further work.

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